Research proposal for Ph. D. programme

Proposed Title: On the ideal elements of le-semigroups

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In the ideal theory of commutative rings, it was observed by W. Krull [6] that several results do not depend on the fact that the ideals are composed of elements. Same is true for the ideal theory of semigroups also. Consequently, these results can be formulated in a more general setting of lattice-ordered semigroups where an element represents an ideal of the ring or semigroup as an undivided entity. There are series of articles dealing with lattice-ordered semigroups, generalizing theorems from commutative ideal theory [1], [2], [3] and from ideal theory of semigroups [4], [5], [7], [8]. Presently lattice ordered semigroups are providing us a general setting not only for abstract ideal theory, but also for order preserving transformations of a finite chain, power semigroups of an arbitrary semigroup and for many other areas of algebra where the objects form similar kinds of lattice-ordered semigroups.

An *ordered semigroup* S is a partially ordered set $(S, \leq)$, and at the same time a semigroup $(S, \cdot)$ such that $\forall a, b, c \in S$

$$a \leq b \ \Rightarrow \ ca \leq cb \text{ and } ac \leq bc.$$

It is denoted by $(S, \cdot, \leq)$. For a nonempty subset H of an ordered semigroup S denote

$$(H] = \{t \in S : t \leq h, \ for \ some \ h \in H\}.$$

A nonempty subset I of an ordered semigroup S is called a *left (right) ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is called an *ideal* of S if it is both a left and a right ideal of S. An ordered semigroup S is called *left (right) simple* if it has no non-trivial proper left (right) ideal. Similarly we define simple ordered semigroups. S is called *t-simple* ordered semigroup if it is both left and right simple. Thus an ordered semigroup S is t-simple if and only if for every $a, b \in S$, there are $x, y \in S$ such that $a \leq xb$ and $a \leq by$. In the theory of ordered semigroups t-simple semigroups plays the roll same as groups in the theory of semigroups (without order).

An ordered semigroup $(S, \cdot, \leq)$ is said to be a *lattice ordered semigroup* if $(S, \leq)$ is a lattice. If moreover $(S, \leq)$ has a greatest element which is usually denoted by e, then the ordered semigroup $(S, \cdot, \leq)$ is called an *le-semigroup*. It is easy to verify that in an le-semigroup $(S, \cdot, \leq)$ for all $a, b, c \in S$,

$$a(b \vee c) = ab \vee ac \text{ and } (a \wedge b)c = ac \wedge bc.$$

For different examples and relevance, both classical and modern, of the *le*-semigroups we refer to [7].

Two mappings l and $r : S \longrightarrow S$ are defined by: $l(x) = ex \vee x$ and $r(x) = xe \vee x$. An $x \in S$ is said to be a *left ideal [right ideal] element* if $l(x) = x$ [respectively, $r(x) = x$]. One can easily verify that for every $x \in S$, the elements $l(x)$ and $r(x)$ are respectively the least left and right ideal elements above x. For short, we say that they are respectively the left and right ideal elements *generated* by x respectively. These notions of ideal elements were motivated strongly by the ideal theory of commutative rings. Though initially commutative ideal theory was the main guiding force to the theory of *le*-semigroups but now it is taking ideas from many other branches of mathematics where there is a notion of order.

In our proposed thesis we would like to take advantage of recent developments in both commutative ideal theory and the theory of order preserving transformations of a finite chain. We set our problem to give a general setting to the recent developments of these theories using *le*-semigroups.

References

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